Some Characteristic of $\alpha -$Fuzzy Orders Relative to $\alpha -$Fuzzy Subgroups , Normal Subgroup and $\alpha -$Fuzzy Cyclic Group

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ABSTRACT

In this communication of the paper depicted the $\alpha - FOs$ of a group and then explain the idea of $\alpha - FSG$ and $-FNSG$. More over , we generalized $\alpha - FOs$ relative to $\alpha -$cyclic group and investigate some characteristic of related algebraic results.

Keywords

Fuzzy Set (FS), fuzzy subset (FSb) ,fuzzy orders (FO), fuzzy group (FG),fuzzy subgroup(FSG), $\alpha -$fuzzy orders $(\alpha - FO)$, $\alpha -$fuzzy group$(\alpha - FG)$, $\alpha -$fuzzy subgroup$(\alpha - FSG)$, $\alpha -$fuzzy normal subgroup$(\alpha - FNSG)$ and $\alpha -$Cyclic group.

1. INTRODUCTION

Zadeh L A[9] explored the new idea of fuzzy subsets of a nonempty set in 1965. Abou-Zaid S[1] ,introduced the characteristic fuzzy subgroups of a finite group in 1991. Rosenfeld A[8], explored the new concept of fuzzy groups in 1971. In 1984, Fuzzy Normal subgroups and fuzzy cosets derived from Mukherjee N P and Bhattacharya P[7]. Liu W J[5], described the new idea of fuzzy invariant subgroups and fuzzy ideals in 1982. In 1994, introduced the new notion of fuzzy orders relative to fuzzy subgroups by Jae-Gyeom Kim[6]. In 1981, produced the new concept of fuzzy groups and level subgroups in Das P S[4]. Asaad M[3], developed the new idea in groups and fuzzy subgroups in 1991. In 1988, Some properties of fuzzy groups in explored from the idea is Akgul M[2].

In this research paper arranged as that, section 2 basic fundamental elementary definition and related the results which are through this research article. In section 3, we have define α −fuzzy orders with respect to the α −fuzzy subgroups and α −fuzzy normal subgroups described the some algebraic characteristic results and section 4, we will be introduced the α −fuzzy orders with respect to the α −fuzzy cyclic group and their some generalization results explained.

2. PRELIMINARIES

Definition: 2.1[9]

Let X be a non-empty set . A FSb of the set X is a mapping μ : X→ [0, 1].

Definition: 2.2[8]

Let be a group. A of is a of if

(i) $\mu(xy) \geq min\{\mu(x), \mu(y)\}$

(ii) $\mu(x^{-1}) \geq \mu(x)$, for all $x, y \in G$.

(iii) From this definition, we clearly have $\mu(x^{-1}) = \mu(x)$, for all $x, y \in G$.

Definition: 2.3[6]

Let G be a group. A FSG μ of G is normal (Invariant) in G if $\mu(xy) = \mu(yx)$ for all $x, y \in G$.

Theorem: 2.4[8]

Let be a group and let be a of . Then

(i) $\mu(x) \leq \mu(e)$, for all $x, y \in G$.

(ii) $if \mu(xy^{-1}) = \mu(e)$, then $\mu(x) = \mu(y)$

Theorem: 2.5 [7]

Let G be a group and let μ be a FSG of G. Then μ is normal in G if and only if $\mu(y^{-1}xy) = \mu(x)$, for all $x, y \in G$.

Theorem: 2.6 [4]

Let be a cyclic group of order , where is a prime number. If is a of , then for all $x, y \in G$.

(i) If $O(x) > O(y), then\ \mu(x) \leq \mu(y)$.

(ii) If ( ) = ( ), then ( ) = ( ).

Theorem: 2.7 [2]

Let be a finite group and let be a of . Then

(i) $\mu(x^K) \geq \mu(x)$ for any integer K and for all $x \in G$.

(ii) If $O(x)/O(y)$, then $\mu(y) \leq \mu(x)$ for $x, y \in \langle Z \rangle$, where $z \in G$.

(iii) If $(O(x), K) = 1$, then $\mu(x^k) = \mu(x), where\ k \in Z\ and\ x \in G$.

Theorem: 2.8

Let G be a group. For $x, y, z \in G$, we have

(i) If $x^m = e$, then $O(x)/m$, where $m \in Z$.

(ii) $O(x^m) = O(x)/(O(x), m)$, where $m \in Z$.

(iii) If $(O(x), O(y)) = 1$ and $xy = yx$, then $O(xy) = O(x) \times O(y)$.

(iv) If $z = y^{-1}xy$, then $O(z) = O(x)$.

(v) If $O(z) = mn$ with $(m, n) = 1$, then $z = xy = yx$ for some $x, y \in G$ with $O(x) = m$ and ( ) = . Further, such an expression for is unique.

Definition: 2.9 [6]

Let μ be a FSG of a group G. For a given $x \in G$, the least positive integer n such that $\mu(x^n) =$ ( ) is the of with respect to [briefly, ( )]. If no such exists, is of infinite with respect to .

3. SOME CHARACTERISTIC OF $\alpha - FOs$ RELATIVE TO $\alpha -$

Definition: 3.1

Let A^α be a $\alpha - FSG$ of a group G. For a given $\theta \in G$, the least positive integer n such that $A^\alpha(\theta^n) = A^\alpha(e)$ is the $\alpha - FO$ of θ with respect to A^α [briefly, $FO_{A^\alpha}(\theta)$]. If no such n exists, θ is of infinite $\alpha - FO$ with respect to A^α.

$\therefore O(\theta)$ and $O(\varphi)$ does not imply that of $FO_{A^\alpha}(\theta)$ and $FO_{A^\alpha}(\varphi)$,

Example: 3.1.1

Let $G = \{a, b / a^2 = b^2 = (ab)^2 = e\}$ be the Klein four-group. Define a $\alpha - FSG$ A^α of G by ( ) = ( ) = and ( ) = ( ) = $_1$, where > $_1$. Clearly, ( ) = ( ) = 2, but ( ) = 2 and ( ) = 1.

Proposition: 3.2

Let A^α be a $\alpha - FSG$ of a group G. For $\theta \in G$, if $A^\alpha(\theta^m) = A^\alpha(e)$ for some integer m, then ( )/ .

Proof:

Let $FO_{A^\alpha}(\theta) = n$. If ∃ integers $s \text{ and } t\ :\ m = ns + t$, where $0 \leq t < n$.
Then, $A^\alpha(\theta^t) = A^\alpha(\theta^{m-ns}) = A^\alpha(\theta^m(\theta^n)^{-s}) \geq min\{A^\alpha(\theta^m), A^\alpha((\theta^n)^{-s})\}$
$\geq min\{A^\alpha(e), A^\alpha(\theta^n)\} = min\{A^\alpha(e), A^\alpha(e)\} = A^\alpha(e)$.
Hence $t = 0$, by the choice of n. If $O(\theta)$ is finite then $FO_{A^\alpha}(\theta)$ is clearly finite for all $\alpha - FSG\ A^\alpha$ of G. If $O(\theta)$ is infinite, then for each positive integer n, ∃ a $\alpha - FSG\ A^\alpha{}_n$ of G : $FO_{A^\alpha}\ (\theta) = n$ as follows. ■

Example: 3.2.1

Let be an element of infinite order in the group . For each positive integer , define the $\alpha - FSG\ A\ {}_n$ of G by $A\ {}_n(\varphi) = \begin{cases} t_o\ if\ \varphi \in \langle\theta^n\rangle, \\ t_1\ otherwise, \end{cases}$

Where $t_o > t_1$. Clearly, $FO_{A^\alpha{}_n}(\theta) = n$. ■

Corollary: 3.2.2

Let A^α be a $\alpha - FSG$ of a group G. Then $FO_{A^\alpha}(\theta)/\ O(\theta)$ for all $\theta \in G$.

Proposition: 3.3

Let A^α be a $\alpha - FSG$ of a group G, and let θ and φ be elements of G : $(FO_{A^\alpha}(\theta), FO_{A^\alpha}(\varphi)) = 1$ and $\theta\varphi = \varphi\theta$. If $A^\alpha(\theta\varphi) = A^\alpha(e)$, then $A^\alpha(\theta) = A^\alpha(\varphi) = A^\alpha(e)$.

Proof:

Let $FO_{A^\alpha}(\theta) = n$ and $FO_{A^\alpha}(\varphi) = m$. Then $A^\alpha(e) = A^\alpha(\theta\varphi) \leq A^\alpha((\theta\varphi)^m) = A^\alpha(\theta^m\varphi^m)$.
Thus ( ) = ( )= ( ). Therefore, / , by pro.. (3.2). But (,
) = 1. Thus = 1, i.e., ( ) = ( ).
Hence $A^\alpha(\varphi) = A^\alpha(\theta) = A^\alpha(e)$. ■

Within the proposition, although is normal, the belief = may not be omitted .

Corollary: 3.3.1

Let A^α be a $\alpha - FSG$ of a group G, and let θ and φ be elements of G such that (( ), ( )) = 1 and = . If ( ) = ( ), then ( ) = (y)= ( ).

Neither the assumption $(FO_{A^\alpha}(\theta), FO_{A^\alpha}(\varphi)) = 1$ in pro…(3.3) nor the assumption $(O(\theta), O(\varphi)) = 1$ in corollary 3.3.1 can be omitted. In fact, in example 3.1.1 $A^\alpha(a) = A^\alpha(b) \neq A^\alpha(e)$, but $FO_{A^\alpha}(a)$=$FO_{A^\alpha}(b) = O(a) = O(b) = 2$. ■

Theorem: 3.4

Let A^α be a $\alpha - FSG$ of a group G. Let $FO_{A^\alpha}(\theta) = n$, where $\theta \in G$. If m is an integer with = (,), then ( ) = / .

Proof:

Let ( ) = . First we have
(( ))‾= ( ) for some integer
$\geq A^\alpha(\theta^n) = A^\alpha(e)$.
Thus $t/n/d$ by pro..(3.2). Because $d = (m, n)$, ∃ integer i and $j : ni + mj = d$.
We the have

( ) = ((+)) = ( )
$\geq min\{A^\alpha((\theta^n)^{ti}), A^\alpha((\theta^m)^t)^j)\}$
$\geq min\{A^\alpha(\theta^n), A^\alpha((\theta^m)^t)\}$
= { ( ), ( )} = ( ).

This implies that n/td i.e., $n/d/t$. Consequently , $t = n/d$. ■

Proposition: 3.5

Let A^α be a $\alpha - FSG\ G$. Let $FO_{A^\alpha}(\theta) = n$, where $\theta \in G$. If m is an integer with $(n, m) = 1$, then ( ) = ( ).

Proof:

Because $(n, m) = 1$, ∃ integers s and $t : ns + mt = 1$.
We then have

( ) = (+) = (( ))( )))
$\geq min\{A^\alpha(\theta^n)^s), A^\alpha(\theta^m)^t)\}$
$\geq min\{A^\alpha(\theta^n), A^\alpha(\theta^m)\}$
= { ( ), ( )}
$= A^\alpha(\theta^m) \geq A^\alpha(\theta)$. ■

Theorem: 3.6

Let A^α be a $\alpha - FSG$ of a group G. Let $FO_{A^\alpha}(\theta) = n$, where $\theta \in G$. If $i \equiv j(mod\ n)$, where $i, j \in Z$, then $FO_{A^\alpha}(\theta^i) = FO_{A^\alpha}(\theta^j)$.

Proof:

Let ( ) = and ( ) = . By the assumption, = + for some integer .
Now, (( )) = ((+)) = (( ) ( )))

$\geq \ min\{A^{\alpha}(\theta^{j})^{s}),\ A^{\alpha}(\theta^{n})^{ks})\}$

$\geq min\{A^{\alpha}(e), A^{\alpha}(\theta^{n})\}$

= { ( ), ( )},

And so t/s. Similarly, s/t. Thus we have $t = s$. ■

Theorem: 3.7

Let A^{α} be a $\alpha - FSG$ of a group G, and let θ and φ be elements of $G : \theta\varphi = \varphi\theta$ and $(FO_{A^{\alpha}}(\theta), FO_{A^{\alpha}}(\varphi)) = 1$. Then $FO_{A^{\alpha}}(\theta\varphi) = FO_{A^{\alpha}}(\theta) \times FO_{A^{\alpha}}(\varphi)$.

Proof:

Let ( ) = , ( ) = and ( ) = .

Then (( )) = ( )

$\geq min\{A^{\alpha}((\theta^{s})^{t}), A^{\alpha}((\varphi^{t})^{s})\}$

$\geq min\{A^{\alpha}(\theta^{s}), A^{\alpha}(\varphi^{t})\}$

= { ( ), ( )} = ( ).

Thus / ,Now ( ) = (( )) = ( ). Besides,

(( ), ( )) = 1 .

$\therefore A^{\alpha}(\theta^{n}) = A^{\alpha}(\varphi^{n}) = A^{\alpha}(e)$ both s and t divide n .

$\therefore st/n$, because $(s, t) = 1 \Rightarrow n = st$. ■

Corollary: 3.7.1

Let A^{α} be a $\alpha - FSG$ of a group G, and let θ and φ be elements of $G : \theta\varphi = \varphi\theta$ and $(O(\theta), O(\varphi)) = 1$. Then $FO_{A^{\alpha}}(\theta\varphi) = FO_{A^{\alpha}}(\theta) \times FO_{A^{\alpha}}(\varphi)$.

∵ supposing A^{α} is normal subgroup, the assumption $\theta\varphi = \varphi\theta$ may not be omitted.

Example: 3.7.2

Define a $\alpha - FNSG\ \ A^{\alpha}$ of the symmetric group S_4

( ) = { = , t_1 *otherwise*,

Where > 1. Now, let = (1 2) and = (2 3 4). Then ( ) = 2, ( ) = 3, $FO_{A^{\alpha}}(\theta\varphi) = FO_{A^{\alpha}}(\varphi\theta) = 4$, and $\theta\varphi \neq \varphi\theta$. ■

Theorem: 3.8

Let A^{α} be a $\alpha - FSG$ of a group G. For $z \in G$, if $FO_{A^{\alpha}}(z) = nm$ with $(n, m) = 1$, then $\exists\ \theta$ and φ in $G : z = \theta\varphi = \varphi\theta$, $FO_{A^{\alpha}}(\theta) = m$ and $FO_{A^{\alpha}}(\varphi) = n$. Furthermore explain for z is unique in the sense of α −fuzzy grades, i.e., if (θ, φ) and (θ_1, φ_1) are such pairs, then $A^{\alpha}(\theta) = A^{\alpha}(\theta_1)$ and ( ) = (1).

Proof

Because $(m, n) = 1$, ∃ integers s and $t :\ ms + nt = 1$.

Here (,) = (,) = 1. Let = and = . Then = = , and by theorem 3.4, ( ) = ( ) = and ( ) = ( ) = . This proves the existence of and .

Let (,) and (1, 1) be pairs satisfied.

since ( ) = (1) = and ( ) = (1) = ,

$\Rightarrow A^{\alpha}(\theta) = A^{\alpha}(\theta^{1-ms}) = A^{\alpha}(\theta^{nt}) = A^{\alpha}(\theta^{nt}\varphi^{nt}) = A^{\alpha}((\theta\varphi)^{nt})$

= ((1 1)) = (1 1)= (1)

$= A^{\alpha}({\theta_1}^{1-ms}) = A^{\alpha}(\theta_1)$.

Similarly, ( ) = (1).

This proves the uniqueness of (θ, φ). ■

Theorem: 3.9

Let A^{α} be a $\alpha - FNSG$ of a group G. Then $FO_{A^{\alpha}}(\theta) = FO_{A^{\alpha}}(\varphi^{-1}\theta\varphi)$ for all $\theta, \varphi \in G$.

Proof:

Let $\theta, \varphi \in G$, then we have $A^{\alpha}(\theta^{n}) = A^{\alpha}(\varphi^{-1}\theta^{n}\varphi) = A^{\alpha}((\varphi^{-1}\theta\varphi)^{n})$ for all $n \in Z$.

Thus $FO_{A^{\alpha}}(\theta) = FO_{A^{\alpha}}(\varphi^{-1}\theta\varphi)$. ■

$\therefore A^{\alpha}$ is not normal in G.

Example: 3.9.1

Let D_3=$\{a, b/a^{3} = b^{3} = e, ba = a^{2}b\}$ be the group with 6 elements. Define a $\alpha - FSG\ A^{\alpha}$ of 3 by

$$A^{\alpha}(\theta) = \{ \begin{matrix} t_o\ if\ \theta \in \langle b \rangle, \\ t_1\ \ otherwise \end{matrix}$$

Where $t_o > t_1$. Then $a^{-1}ba \notin \langle b \rangle$, and so $FO_{A^{\alpha}}(b) = 1 \neq FO_{A^{\alpha}}(a^{-1}ba)$. ■

4. ALGEBRAIC PROPERTIES OF $\alpha - FOs$ IN A CYCLIC GROUP

Lemma: 4.1

Let A^{α} be a $\alpha - FSG$ of a cyclic group G and let a and b be any two generators of G. Then ( ) = ( ).

Proof

We have apply for Theroem..(3.4).

Theorem: 4.2

Let A^{α} be a $\alpha - FSG$ of a cyclic group G of finite order n. Then, $\forall\ \theta, \varphi \in G$;

(i) If ( ) = ( ), then ( ) = ( ).

(ii) If ( )/ ( ), then ( )/ ( ).

(iii) If $O(\theta) > O(\varphi)$, then $FO_{A^{\alpha}}(\theta) \geq FO_{A^{\alpha}}(\varphi)$.

Proof

Let $G = \langle a \rangle$. Let $\theta = a^{s}$, $\varphi = a^{t}$, and $FO_{A^{\alpha}}(a) = m$.

is a specific generator of .

Then ( ) = /(,), ( ) = /(,), ( ) = /(,) and / ,

(i) Follows from (ii).

(ii) If ( )/ ( ), then (,)/(,), and so (,)/(,), because / . Thus ( )/ ( ).

(iii) If $O(\theta) > O(\varphi)$, the $(s,n) < (t,n)$, and so $(s,m) \leq (t,m)$, because m/n. Thus $FO_{A^{\alpha}}(\theta) \geq FO_{A^{\alpha}}(\varphi)$. ■

Theorem: 4.3

Let A^{α} be a $\alpha - FSG$ of a cyclic group G of finite order. Then , $\forall\ \theta, \varphi \in G$:

(i) If ( ) = ( ), then ( ) = ( ).

(ii) If $FO_{A^{\alpha}}(\theta)/FO_{A^{\alpha}}(\varphi)$, then $A^{\alpha}(\theta) \geq A^{\alpha}(\varphi)$.

Proof

Let $G = \langle a \rangle$. Let $\theta = a^{s}$, $\varphi = a^{t}$, and $FO_{A^{\alpha}}(a) = m$.

is a specific generator of .

Then ( ) = /(,) and ( ) = /(,),

Let $s = h(s,m)$, $t = i(t,m)$ and $m = j(t,m) = k(s,m)$ for some $h, i, j, k \in Z$.

If $FO_{A^{\alpha}}(\theta)/FO_{A^{\alpha}}(\varphi)$, then $(t,m)/(s,m)$. So $t/si = h(s,m)i$ and $m/sj = h(s,m)j$,

$\Rightarrow \quad A^{\alpha}(\theta) = A^{\alpha}(a^{s})$

$= A^{\alpha}(a^{s(iv+jw)})$ for some $u, w \in Z$, since $(i,j) = 1$

$= A^{\alpha}(a^{siv}a^{sjw}) \geq min\{A^{\alpha}(a^{siv}), A^{\alpha}(a^{sjw})\}$

$\geq min\{A^{\alpha}(a^{t}), A^{\alpha}(a^{m})\} = min\{A^{\alpha}(\varphi), A^{\alpha}(e)\} = A^{\alpha}(\varphi)$. ■

Corollary: 4.3.1

Let A^{α} be a $\alpha - FSG$ of a cyclic group G of finite order. Then, $\forall\ \theta, \varphi \in G$:

(i) If $O(\theta) = O(\varphi), then\ A^{\alpha}(\theta) = A^{\alpha}(\varphi)$.

(ii) If $O(\theta)/O(\varphi), then\ A^{\alpha}(\theta) \geq A^{\alpha}(\varphi)$. ■

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