Decomposition of Graphs using Prime Labeling Algorithm

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Abstract

A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. Let G(V, E) be a graph with vertex set V = V(G) and edge set E = E(G). A prime labeling is a bijective function f : V(E) → {1, 2, …, n} such that the greatest common factor of {f(u), f(v)} = 1 for each e = uv in E(G). In this paper we introduce maximal prime labeling and maximal composite labeling. The concept of maximal prime labeling and maximal composite labeling helps to decompose complete graphs. Also we establish an algorithmic approach for decomposition of graphs using prime labeling.

1. Introduction

The graph labeling in graph theory has a fast development recently. In most applications labels are positive or nonnegative integers, through in general real numbers could be used. Graph labeling‟s are useful family of mathematical models for a broad range of applications like cryptography, coding theory, communication networks and data security, etc. The notation of a prime labeling orginated with Entringer and was introduced in a paper in Taut, Dabboucy and Howalla [1], [3]-[6].

Let G(V, E) be a graph with vertex set V = V(G) and edge set E = E(G). A prime labeling is a bijective function f : V(E) → {1, 2, …, n} such that the greatest common factor of {f(u), f(v)} = 1 for each e = uv in E(G). A graph acknowledging a prime labeling is called a prime graph [2], [7].

Example: Prime labeling of the ladder graph $L_7 = P_2 \times P_7$ is shown in figure 1.

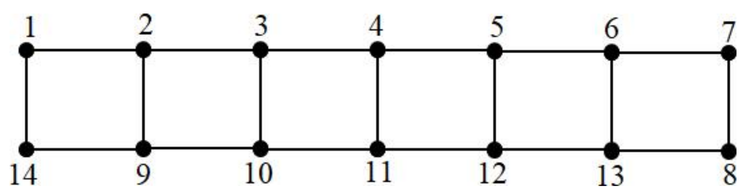

Figure 1

A graph G is decomposable into the subgraphs $G_1, G_2, ..., G_n$ of G, if no G_i , i = 1,2,...,n, has isolated vertices and the edge set of G can be partitioned into the subsets $E(G_1), E(G_2), ..., E(G_n)$. We say that there is a packing of $G_1, G_2, ..., G_k$ into the complete graph K_n if there exist injective mappings $\alpha_i : V(G_i) \rightarrow V(K_n)$, i = 1,2,...,k such that $\alpha_i^*(E(G_i)) \cap \alpha_j^*(E(G_j)) = \varphi$ for $i \neq j$, where the map $\alpha_i^*: E(G_i) \rightarrow E(K_n)$ is induced by α_i.

Similarly, suppose G is a graph of order m and H is a graph of order $n \geq m$ and there exists an injection $\alpha : V(G) \rightarrow V(H)$ such that $\alpha^*(E(G)) \cap \alpha^*(E(H)) = \varphi$, then we say that there is a packing of G into H, and in case n = m, then we say that there is a packing of G and H or H and G are packable.

Clearly, decomposition and packing of graphs are inverse processes. Here, we obtain results on decomposition of graphs and from these it is easy to obtain corresponding results on packing [8], [9]. In this paper we introduce maximal prime labeling and maximal composite labeling. The concept of maximal prime labeling and maximal composite labeling helps to decompose complete graphs. Also we establish an algorithmic approach for decomposition of graphs using prime labeling.

2. Maximal prime graph

In this section we introduce the concept of maximal prime labeling of a graph G.

Definition 2.1:

A maximal prime graph G of order n is a simple graph with vertex set {1, 2, 3, …, n} and any two vertices i and j are adjacent if and only if $GCD(i, j) = 1, i \neq j$.

Example 2.2: Maximal prime graph G of order G of order 8 is shown in figure 2.

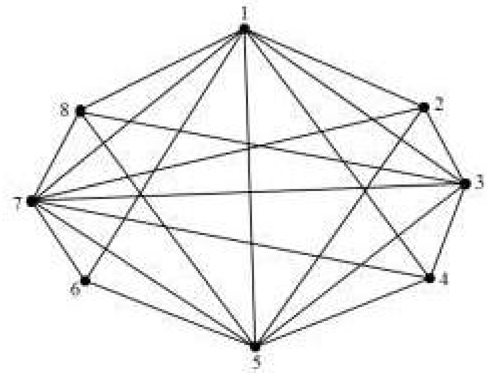

Figure 2

Theorem 2.3:

Any simple graph H of order n is a prime graph if and only of it is a spanning subgraph of the maximal prime graph of order n.

Proof: Let $G(V,E)$ be the maximal prime graph of order n. Suppose $H(V_{11}, E_{11})$ is a prime graph with n vertices. Then there exists an injective map $f : V_{11} \to \{1, 2, \ldots, n\}$ and hence $V_{11} = V$. Also for any edge $xy \in E_{11}, GCD[f(x), f(y)] = 1$. Therefore $xy \in E$. Thus H is a spanning subgraph of G.

Conversely, if H is a spanning subgraph of G, then $V_{11} = V$ and $E_{11} \subseteq E$ which imply that every edge $xy \in E_{11}$ will have relative prime integers as its end vertex labels. Hence H is prime.

Around 1980, Rojer Entringer proposed the conjecture that all trees have a prime labeling that still remains unsolved. By theorem 2.3, the above conjecture is equivalent to the following conjecture.

Conjecture 2.4: Every tree of order n is a spanning tree of the maximal prime graph of order n.

Theorem 2.5:

If G is the maximal prime graph of order n, then (i) G is Hamiltonian (that is, G contains a cycle C_n of length n). (ii) For every $v \in V_G$, $deg(v) = n - 1$ if and only if $f(v) = 1$ or $f(v) \in P(n/2, n)$ where $P(t, n)$ is the set of all primes x such that $t < x \leq n$.

Proof: (i) (1, 2, …, n) is a Hamiltonian cycle in G.

(ii) Since I and the elements of $P(n/2, n)$ are the only elements which are relatively prime to all the other elements in the set {1, 2, … n}, the result is proved.

Theorem 2.6:

If G is a maximal prime graph of order n, then $\beta(G) = \lfloor n/2 \rfloor$.

Proof: In a maximal prime graph the vertices labeled with even integers form a maximal independent set and hence the theorem follows.

However the converse is not true. For example the n-cycle C_n for n ≥ 4 is prime (not maximal prime) and $\beta(G) = \lfloor n/2 \rfloor$.

3. Maximal composite graph

In this section we define maximal composite labeling of a graph G.

Definition 3.1:

A labeling of a graph G is called a composite labeling of G if its vertices are labeled with integers 1, 2, …, $|V(G)|$ such that for any edge $e = uv \in E(G), GCD[f(u), f(v)] > 1$. G is called a maximal composite graph of order n if it is the compliment of the maximal prime graph of order n.

Example 3.2: The maximal composite graph of order 8 is shown in figure….

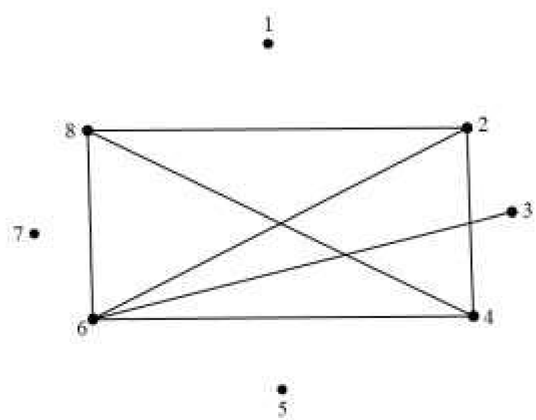

Figure 3

Theorem 3.3:

If G is a maximal composite graph of order n, then $(\ ) = |\ (1,\ )| + 1$.

Proof: If G is the maximal composite graph of order n, then the vertex labeled 1 is an isolated vertex. The vertex labeled 2 is one of the vertices of the clique formed by the vertices labeled with even numbers. The vertices labeled with any other prime will be either isolated vertex (if it is $>\ /2$). Hence $(\ ) = |\ (1,\ )| + 1$.

Corollary 3.4: For any composite graph G, $(\ ) = |\ (1,\ )| + 1$.

Corollary 3.5: The clique graph K(G) of a maximal composite graph G has exactly $|P(1,n)| + 1$ vertices in which $|P({}^{n}/_{2}, n)| + 1$ vertices are isolates.

4. Prime labeling Algorithm

Here we established an algorithm approach using maximum prime graph of order „n" and maximal composite graph of order „n".

```
INPUT:  N, Number of Vertices
OUTPUT:     A, number of relative prime pairs
            B, number of non-relative prime pairs
BEGIN
    READ N
    LET A=0 and B =0
    FOR Z=1 to N-1 STEP 1 DO
        FOR K =Z+1 to N STEP 1 DO
            IF GCD(Z,K) = = 1 THEN
                PRINT „1"
                A= A+1
            ELSE
                PRINT „0"
                B=B+1
            END IF
        END FOR
    END FOR
    PRINT A, B
END
```

Output

Enter the Number of vertices: 10

Vertex	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1
2		1	0	1	0	1	0	1	0
3			1	1	0	1	1	0	1
4				1	0	1	0	1	0
5					1	1	1	1	0
6						1	0	0	0
7							1	1	1
8								1	0
9									1

A B

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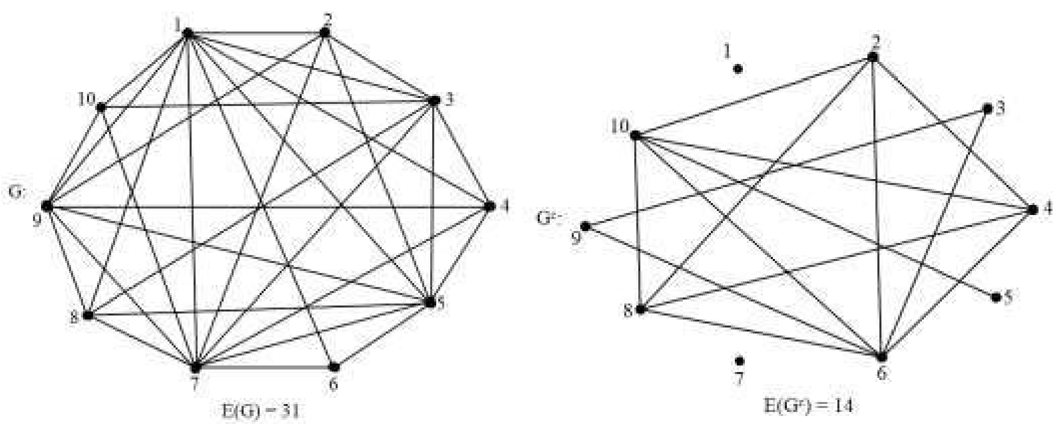
31 14

From the above algorithm we could noticed that the concepts of maximal prime graph and maximal composite graph are used to decompose complete graphs. The output graph of the above algorithm is given below.



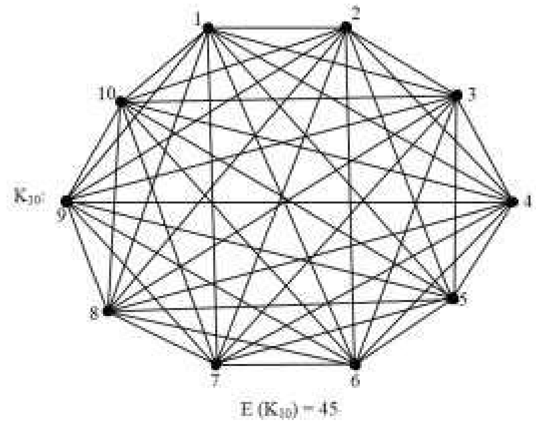


Figure 5

Conclusion:

This concludes that maximal prime labeling and maximal composite labeling helps to decompose complete graphs. Also use the prime labeling algorithm to mold the complete graph. We can apply the algorithm in the field of decomposition related work.

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