Quantitative Method of Singularly Perturbed

Delay Differential Equations

Dr Dinesh Kumar

Abstract – In this paper, singularly perturbed delay differential equation is solved by converting to first order ordinary differential equation and using two- point Gaussian quadrature rule to approximate the solution. Numerically the values of the derivatives and the integrals are approximated to get tridiagonal system. Thomas algorithm is used to solve the tridiagonal system of the method. Convergence of the proposed method is analyzed
.

Keywords: singularly perturbed delay differential equation, Taylor series, Boundary Layer,

Tridiagonal system

I. Introduction

Singularly perturbed differential-difference equations (SPDDEs) arise very frequently in the mathematical modeling of real life situations in science and engineering variational problems in control theory[2],[6],[14]. In general, any ordinary differential equation in which the highest derivative is multiplied by a small positive parameter and containing at least one delay/advance parameter is known as singularly perturbed differential difference equation. Lange and Miura [14]-[18] published a series of papers extending the method of matched asymptotic expansions initially developed for ordinary differential equations to obtain approximate solution of singularly perturbed differential-difference equations. Numerical analysis of singularly perturbed differential - difference turning point problems was initiated by Kadalbajoo and Sharma. In a series of papers, see [8]-[10], they gave many robust numerical techniques for the solution of such type of problems. Kadalbajoo and Sharma [8] elucidate a numerical method to solve boundary value problems for singularly perturbed differential difference equation of the form: , under the interval and boundary conditions on and on . Kadalbajoo and Sharma [9] proposed a numerical method to solve boundary-value problems for a singularly perturbed differential-difference equation of a mixed type, i.e. which contains both types of terms having negative shifts as well as positive shifts, and considered the case in which the solution of the problem exhibits rapid oscillations. Kadalbajoo and Sharma [10] described a numerical approach based on finite difference method to solve a mathematical model arising from a model of neuronal variability. Patidar and Sharma [20] combined fitted-operator methods with Micken's non-standard finite difference techniques for the numerical approximations of singularly perturbed linear delay differential equations. Kadalbajoo et al. [11] derived ϵ-uniformly convergent fitted methods for the solution of singularly perturbed differential difference equation (SPDDE). Kumar and Sharma [13] presented a numerical scheme based on B-spline collocation to approximate the solution of boundary value problems for singularly perturbed differential-difference equations with delay as well as advance. Amiraliyev and Cimen [5] derived a numerical method for singularly perturbed boundary value problem for a linear second order delay differential equation with a large delay in the reaction term. The authors constructed an exponentially fitted differential scheme on a uniform mesh accomplished by the method of integral identities with the use of exponential basis functions and interpolating quadrature rules with weight and the remainder term in the integral form.

II. Description of the problem

Consider a linear singularly perturbed two -point boundary value problem of the form: $\epsilon y''(x) + a(x) y'(x) + b(x) y(x) = f(x)$, $0 \leq x \leq 1$, (1) on (0, 1), under the boundary conditions $y(x) = \phi(x)$ on $-\delta \leq x \leq 0$,

$$(2) \quad y(1) = \gamma ,$$

where ϵ is small parameter, $0 < \epsilon \ll 1$ and δ is also delay

parameter $0 < \delta \ll 1$ a( ),b( ),x x $f(x)$,and $\phi(x)$ are bounded continuous functions in (0,1) and γ is a finite constant. For $\epsilon \rightarrow 0$ the solution of the boundary value problem (1) and (2) exhibits layer or oscillatory behavior depending on the sign of $a(x) - b(x)$. If

$a(x) - b(x) \geq M > 0$, where M is a positive constant, the solution of the problem (1) and (2) exhibits layer behavior and if $a(x) - b(x) \leq M < 0$, it exhibits oscillatory behavior. The boundary value Problem considered here is of the reaction-diffusion type, therefore if the solution exhibits layer behavior, there will be two boundary layers which will be at both end point i.e., at $x = 0$ and $x = 1$.

III. Numerical Method

We divide the interval $[0,1]$ into N equal parts of mesh size h, with $0 = x_0, x_1, ..., x_{N-1}, x_N = 1$ being the mesh points. Hence, we have $x_i = ih$, for $i = 0$ to N. We choose n such that $x_n = \dfrac{1}{2}$. In the

Interval $[0, 2]$,the boundary layer will be at the left

hand side i.e., at $x = 0$ and in $[\frac{1}{2}, 1]$ the boundary

layer will be at the right hand side i.e., at $x = 1$. Therefore we calculate the solution for three Different

intervals $[0, \frac{1}{2}]$, $[\frac{1}{2}, 1]$ and $[\frac{1}{2}, 1]$.

III.1 Left – End Layer   For left hand layer

i.e., $0 \le x \le \frac{1}{2}$, by using Taylor's expansion for $y(x - \Delta)$, we can

say that, $y(x-\Delta) = y(x) - \Delta y'(x)$
Hence,

$$\Delta y'(x) = y(x) - y(x - \Delta)$$

By using this approximation in (1) we get,

$$y'(x) + \epsilon y''(x-\Delta) + a(x)y(x) + b(x)y(x) = f(x)$$
(3)

Inregrating this equation from x to x_i $_{i+1}$ we get,

$$y_{i+1} - y_i + \epsilon \int_{xi}^{xi+1} y''(x-\Delta)dx + \int_{xi}^{xi+1} a(x)y(x)dx + \int_{xi}^{xi+1} b(x)y(x)dx = \int_{xi}^{xi+1} f(x)dx$$
(4)

We use Gause-Legendre quadrature to approximate the value of this integration, for which we need to change the limits to -1 to 1. For this purpose, we introduce a new variable u Where,

$$\frac{2 \cdot x - (x_i + x_{i+1})}{x_{i+1} - x_i}; \quad x = \frac{uh}{2} + \frac{x_i + x_{i+1}}{2}$$

Using this approximation in (4) we get $y_{i+1} - y_i =$

$$y(x_{i+1} - \Delta) = y(x_i - \Delta)$$ (5)

Using two- point Gaussian quadrature we get

$$\int_{-1}^{1} f(x)dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right),$$

For any function f, continuous and differentiable in interval [-1,1].

when $u = \frac{1}{\sqrt{3}}$,

$$\frac{uh + x_i + x_{i+1}}{2} = x_{i+1} - \frac{h\left(1 - \frac{1}{\sqrt{3}}\right)}{2} = x_{i+1} - k,$$

where $k = \frac{h\left(1 - \frac{1}{\sqrt{3}}\right)}{2}$,

$$\frac{1}{\sqrt{3}}, \quad \frac{uh + x_i + x_{i+1}}{2} = x_i + \frac{h\left(1 - \frac{1}{\sqrt{3}}\right)}{2} = u = x_i + k,$$

Where $k = \frac{h\left(1 - \frac{1}{\sqrt{3}}\right)}{2}$

Substituting this in (5) we get $y_{i+1} =$

$$(y_i + y(x_{i+1} - \Delta)) - (y(x_i - \Delta))$$

$$h[\epsilon a(x_{i+1} - k)y(x_{i+1} - \Delta) + k \Delta a(x_i)k y(x_i - \Delta) + k \Delta$$

$$2 [b(x_{i+1} - k)y(x_{i+1} - \Delta)k + b(x_i)k y(x_i - \Delta)k] f(x_{i+1} - k)$$

$$f(x_i - k)]$$

(6)

Now we use following approximations for the y values, $y(x_{i+1} - k - \Delta) = y_{i+1} - (k + \Delta)y_{i+1},$

$$i+1 = \frac{(k + \Delta)(y_{i+1} - y_i)}{h} = y$$

$$k + \Delta$$
$$y_{i+1} - y_{i+1}. \quad h = h$$

Similarly,

$$y(x_i - (k + \Delta)) = \left(1 - \frac{k + \Delta}{h}\right)y_i + \frac{k + \Delta}{h}y_{i+1},$$

$$y(x_{i+1} - k) = \frac{k}{h}y_i + \left(1 - \frac{k}{h}\right)y_{i+1},$$

$$y(x_i - k) = \left(1 - \frac{k}{h}\right)y_i + \frac{k}{h}y_{i+1}$$

By substituting this in (6) we get,

$$h[f(x_{i+1} - k) + f(x_i - k)]$$

$$2$$

$$[h(a(x_i) - k + 2b(x_i) - k)$$

$$2]y_{i+1}$$

$$[2 + k + h + k + k + h] + \frac{h}{2}[a(x_{i+1} - k) + 2a(x_i) - k + 2b(x_{i+1} - k) + 2b(x_i)] = 2 + y_i$$

$$\left[\frac{h}{2}(a(x_{i+1} - k) + 2b(x_{i+1} - k))\right]y_{i+1}.$$

(7)

We can say that,

$$A_i\,y_{i-1} + B_i\,y_i + C_i\,y_{i-1} = b_i,$$

For $1 \le i \le n-1$, where

$$A_i = \left[a(x-k)_i\right] + \left[b(x-k)_i\right]\,h\,\frac{k}{2},$$

$$B_i = \frac{\left[2-\right]}{2}\left[a(x_{i-1}-k)\right] + \frac{k-}{2}\left[a(x_i-k)\right]h\,\frac{h-k-}{}$$

$$\left[b(x_{i-1}-k) - \frac{k}{2} - \left[b(x_i-k)\right]\frac{h-k}{2}\right],$$

$$\left[-h-k\right] - \frac{h-k-}{}\,C_i = \left[a(x_{i-1}-k)\right]$$
$$\left[b(x_{i-1}-k)\right]\frac{}{}, h\ 2 \qquad 2$$
$$h\left[f(x_{i-1}-k) + f(x_i-k)\right]^{b_i}.$$
$$\frac{}{2}$$

III.2 Right – End layer For right hand layer

i.e., $\frac{1}{2} \le x \le 1$, by using Taylor's

expansion for $y''(x-)$ we get, $y''(x$

$-) = y''(x) + \Delta y'''(x)$ Hence,

$\Delta y'''(x) = y''(x +) - y''(x)$

By using this approximation in (1) we get,

$$y(x-) + y(x-) - a(x-)\,y(x) - b(x-)\,y(x) = f(x-) \quad (8)$$

Integrating (8) from x_{i-1} to x_i gives us,

$$y_i - y_{i-1} + \int_{x_{i-1}}^{x_i} y''(x-)dx - \int_{x_{i-1}}^{x_i} a(x-)y(x)dx - \int_{x_{i-1}}^{x_i} b(x-)y(x)dx = \int_{x_{i-1}}^{x_i} f(x)dx$$

(9)

Here the value of u for using Gauss-Legendre Quadrature will be,

$$2x-=x_{i-1}-x_i = uh-x_{i-1}-x_i u \quad -x-$$
$$\qquad x_i-x_{i-1} \quad 2$$

$$u = \frac{1}{\sqrt 3},$$

$$\frac{h-1- + uh-x_{i-1}-x_i-3}{2} -x_i- + \frac{\sqrt{}}{2} -x_i -k,$$

Where $k = \dfrac{h\left[1 + \dfrac{1}{\sqrt 3}\right]}{2},$

At $u = \dfrac{-1}{\sqrt 3},$

$$x_{i-1} - \frac{uh - \dfrac{h-1-}{2} - i-1}{} x-i-1-xi-x - \frac{-\sqrt{}}{} -3- \\ k, \qquad 2$$

Where $k = \dfrac{h\left[1 - \dfrac{1}{\sqrt 3}\right]}{2},$

Substituting this is (8) gives us, y_i

$$-y_{i-1} + y(x-)_i - y(x-)_{i-1}-$$

$$h-\left[a(x-)_i-k)\,y(x-)_i- -k\right]-\left[a(x-)_{i-1}-k)\,y(x-)_{i-1}-k\right]-$$
$$2-\left[b(x-)_i-k)\,y(x-)_i-k\right] - \left[b(x-)_{i-1}-k)\,y(x-)_{i-1}-k\right] = f(x-)_i-k - f(x-)_{i-1}-k-$$

(10)

Now we use following approximations for the y values $y(x-)_i$

$-k-$ and y_i and $-k-$ and y_i,

$$-y_i-, \quad \frac{-k-- -y_{i-1}- y_i-}{h}$$

$$-1- \frac{k--}{h} -y_i- \frac{k--}{h} y_{i-1}.$$

Similarly,

$$y(x-)_{i-1} -k-\quad \frac{}{h}-y_{i-1}- \frac{}{h} \quad -1-k--\quad k-- y_i,$$

$$y(x-)_i-k-\quad \frac{k-}{h} y_i - \frac{k}{h} y_{i-1}, \quad -1-$$

$$\frac{k-}{}$$

$$y(x-)_{i-1}-k- -1- y_{i-1}- k\,yi.$$
$$\qquad h \qquad h$$

By substituting this in (10) we get,

$$hf(x-k)-_i- + f(x-)_{i-1}-k- - h-k-- \quad h-k--$$
$$2 \qquad -h\,a(x-)_{i-1}-k-2-b(x-)_{i-1}-k-2- -y_{i-1}$$

2 h k k h k k

h a x k i 2 a x i1k 2 b x k i 2 b x i1k 2 yi

k k

h a x k i 2 b x k i 2 yi1

We can say that,

$$G y_{i_i 1} \; H y_{i_i} \; I y_{i_i 1} \; b_i, \text{ for } n$$

i N-1, where ,

(11)

G_i a x $i_i 1$ k h k b x $i_i 1$ k h 2 2

2 h k k h k Finally we can write the formulat k ion in the matrix form Hi h a x i k 2 a x i1 k 2 b x i k as, 2 Ay b x bi,1 where k 2

h k h k I_i a x $i_i 1$ k b x $i_i 1$ k h 2 2 h f x i k f x $i_i 1$ k b_i

2

y xn a x n y x n b x n y x n f x n Hn1 In1

(12)

We use following approximations in this equation. y y y x n y x n y x n y_n $\frac{y_{n-1} - y_{n+1}}{2h}$,

$$y x_n \frac{y_{n-1} - 2y_n + y_{n+1}}{h}$$

To get,

$$fn \; n1 \quad 2n n1 \quad an \; yn \; yn-1 2 h y n+1 \; b \; yn \; n$$

h

Rearranging this equation gives us,
(13)

For x_n,

$$fn \frac{h}{2} an \frac{h}{2} an bn yn \frac{h}{2} an \frac{h}{2} an yn$$ For this equation,

We can say that, (14)

Where,

$$Dn yn1 \; En yn \; Fn yn1 \; bn,$$

$$\frac{}{2h} \quad Dn \; 2 \; an , h$$

$$En \; 2 \frac{}{2} a_n \; b_n,$$
h

$$\frac{}{,h} \quad \frac{Fn \; 2 \; a_n}{2h bn \; fn.}$$

$N = 20, 40, 80, 160,$ and 320, for different values of and .

	$N = 100$	$N = 200$	$N = 300$	$N = 400$
0.01	3.157 e-003	1.605e-003	1.075e-003	8.083e-010
0.04	3.143 e-003	1.599e-003	1.075e-003	8.033e-004
0.06	3.078 e-003	1.566e-003	1.050e-003	7.9001e-014

$B1$ C_1 0 ... ... 0 $\frac{1}{2}$

B_2 C_2

A 2 $An1$ $Bn1$ $Cn1$

0

:

$Gn1$

0:

: G_{N-2} H_{N-2} I_{N-2}

0 ... ... 0 $GN1$ $H N1$

y $y1 \cdots yN1$,

b $b1$ $A1y0$ $b2 \cdots bN-2$ $bN-1$ $CN1yN$

IV. Numerical Example

An example of this type of equation is,

$$y \; x \; 2y \; x \; y \; x \; 1, y \; 0 \; 1,$$

A D E F ,
: n n n : x 0 and y 1 1

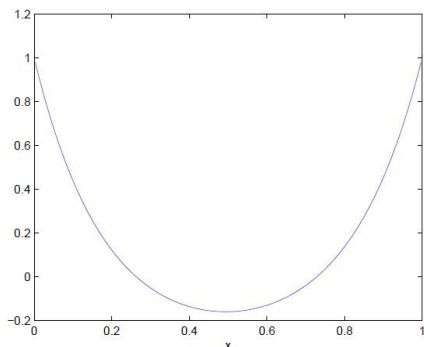this equation, solution was calculated for

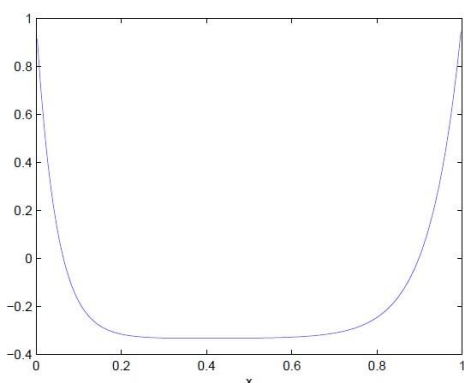Fig.1.Numerical solution for ε 0.1and δ 0.01

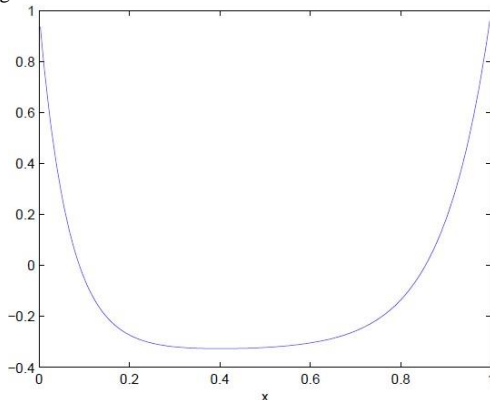Fig.2.Numerical solution for ε 0.01and δ 0.04

Fig.3..Numerical solution for ε 0.02 and δ 0.06

Error term is given by, $u - u_h \le ch^p$,

Where u is the exact solution, u_h is the numerical solution is a constant, h is the mesh size and p is the order of convergence.

Here, as the exact solution is not knows we assume the numerical solution at $N = 320$ to be the exact solution and use other numerical solutions to plot $\log|u - u_h|$ versus $\log h$, for

$\delta = 0.01$ and $\delta = 0.04$,to obtain the convergence plot. Slope of convergence plot is equal to the order of convergence.
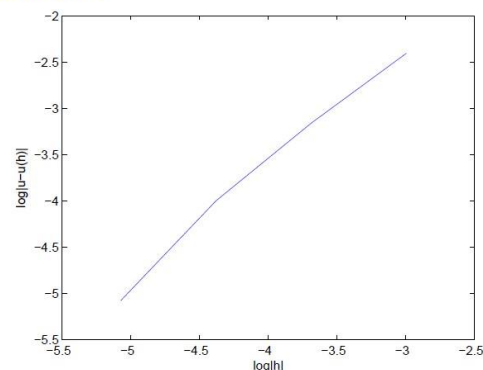

Fig.4. Convergence Plot

From this plot, it can be said that order of convergence for this method is 1.

V. Conclusion

The numerical analysis of singularly perturbed delay differential equation is discussed. In general numerical solution of second order differential equation will be more difficult than numerical solution of first order differential equation. Even though the numerical results are computed at all points of the mesh size h , only few results have been reported. It is observed that the present method approximates the exact solution very well for $h \ge \varepsilon$ for which other classical finite difference methods fail to give good results. The effect of the delay parameter on the solution has also investigated and presented by using graphs. Convergence of the proposed method is analyzed. The present method is simple, very easy to implement and efficient technique for solving singularly perturbed delay differential equations.

References

[1]   E.Angel, R.Bellman, *Dynamic Programming and Partial differential equations* (Academic Press, New York, 1972).

[2]   M.W.Derstine, F.A.H.H.M.Gibbs, and D.L Kaplan, Bifurcation gap in a hybrid optical system, *Phys. Rev. A, V.26,* N.6, 1982, pp. 3720–3722.

[3]   E.P.Doolan, J.J.H. Miller, and W.H.A. Schilders, *Uniform Numerical Methods for problems with Initial and Boundary Layers* (Boole Press, Dublin, 1980).

[4]   L.E.El'sgol'ts, S.B. Norkin, *Introduction to the Theory and Applications of Differential Equations with Deviating Arguments*(Academic Press, New York, 1973).

[5]   M.Gabil, Amiraliyev, Erkan Cimen, Numerical method for a singularly perturbed convection–diffusion problem with delay, *Applied Mathematics and Computation, V.216*, N.8, 2010, pp.2351-2359.

[6]   V.Y. Glizer, Asymptotic analysis and solution of a finite-horizon H1 control problem for singularly-perturbed linear systems with small state delay, *J. Optim. Theory Appl., V.117,* N.2, 2003, pp. 295–325.

[7]   M.K.Kadalbajoo, Devendra Kumar, Fitted mesh B-spline collocation method for singularly perturbed differential–

difference equations with small delay, *Applied Mathematics and Computation*, *V.204*, N.1, 2008, pp.90-98.

[8]   M.K. Kadalbajoo,K.K.Sharma, Numerical analysis of boundary-value problems for singularly perturbed differential-difference equations with small shifts of mixed type, *J. Optim. Theory Appl.*, *V.115*, N.1, 2002, pp. 145-163.

[9]   M.K. Kadalbajoo,K.K.Sharma Numerical analysis of boundary-value problems for singularly perturbed differential-difference equations: small shifts of mixed type with rapid oscillations, *Commun. Numer. Methods Eng.,*, *V.20,* N.3, 2004, pp. 167-182.

[10]   M.K. Kadalbajoo,K.K.Sharma Numerical treatment of mathematical model arising from a model of neuronal variability, *J. Math. Anal. Appl.*, *V.307*, N.2, 2005, pp.606-627.

[11]   M.K. Kadalbajoo,K.K. Sharma,K.C. Patidar, ϵ-Uniformly convergent fitted methods for the numerical solution of the problems arising from singularly perturbed general DDEs, *Appl. Math. Comput.,*, *V.182*, N.1, 2006, pp.119–139.

[12]   H.B.Keller, *Numerical Methods for Two Point Boundary Value Problems*(Blaisdell Publishing Company, 1968).

[13]   V.Kumar, K.K. Sharma K.K, An optimized B-spline method for solving singularly perturbed differential difference equations with delay as well as advance, *Neural, Parallel, and Scientific Computations,*, *V.16*, N.1, 2008, pp.371–386.

[14]   C.G.Lange, R.M. Miura, SingularSingular perturbation analysis of boundary-value problems for differential-difference equations. II. Rapid oscillations and resonances, *SIAM J. Appl. Math.*, *V.45*, N.1, 1985, pp.687–707.

[15]   C.G.Lange, R.M. Miura, perturbation analysis of boundary-value problems for differential-difference equations. III. Turning point problems, *SIAM J. Appl. Math.*, *V.45*, 1985, pp.708–734.

[16]   C.G.Lange, R.M. Miura,, Singular perturbation analysis of boundary-value problems for differential-difference equations. V. Small shifts with layer behaviour, *SIAM J. Appl. Math.*, *V.54*, 1994, pp.249– 272.

[17]   C.G.Lange, R.M. Miura, Singular perturbation analysis of boundaryvalue problems for differential-difference equations. VI. Small shifts with rapid oscillations, *SIAM J. Appl. Math.*, *V.54,* 1994, pp.273–283.

[18]   C.G.Lange, R.M. Miura, Singular perturbation analysis of boundaryvalue problems for differential-difference equations, *SIAM J. Appl. Math.*, *V.42,* 1982, pp.502–531.

[19]   R.E.O'Malley, *Introduction to Singular Perturbations*(Academic Press, New York, 1974).

[20]   K.C. Patidar K.K. Sharma Uniformly convergent non-standard finite difference methods for singularly perturbed differential-difference equations with delay and advance, *Int. J. Numer. Methods Eng.*, *V.66,* N.2, 2006, pp.272-296.

Authors' information

1
 Department of Mathematics, Chsritu Jyoti institute of Technology & Science, Jangaon-506167, Telangana State,India.
diddi.k@gmail.com
 Diddi Kumara Swamy received PhD from National Institution of Technology, Warangal, India (An Institution of National Importance).He is currently Associate Professor in Mathematics in Christu Jyoti Institute of Technology and Science, Jangaon, India.

His research interest on Mathematical modeling and Numerical solutions to Differential difference Equations.
.